

Determine if the sequence $a_n = n \sin \frac{1}{n}$ converges or diverges. If it converges, find its limit.

SCORE: ____ / 5 PTS

Justify your answer using proper algebra and/or calculus. State your conclusion clearly.

$$\lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{x \rightarrow \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{-\frac{1}{x^2} \cos \frac{1}{x}}{-\frac{1}{x^2}}$$

$\uparrow$
 $\infty \cdot 0$
INDETERMINATE

(1/2) (1/2)

$$= \cos 0$$

$$= 1 \quad \text{so } \lim_{n \rightarrow \infty} n \sin \frac{1}{n} = 1$$

(1/2)

SEQUENCE CONVERGES

(1/2)

Determine if the sequence $a_n = \frac{\cos^2 n}{2^n}$ converges or diverges. If it converges, find its limit.

SCORE: ____ / 4 PTS

Justify your answer using proper algebra and/or calculus. State your conclusion clearly.

$$\textcircled{1} \quad 0 \leq \frac{\cos^2 n}{2^n} \leq \frac{1}{2^n} \quad \textcircled{1} \quad \text{SINCE } 0 \leq \cos^2 n \leq 1$$

$$\lim_{n \rightarrow \infty} 0 = \lim_{n \rightarrow \infty} \frac{1}{2^n} = 0 \quad \textcircled{1}$$

$$\text{SO } \lim_{n \rightarrow \infty} \frac{\cos^2 n}{2^n} = 0 \quad \textcircled{\frac{1}{2}}$$

SEQUENCE CONVERGES $\textcircled{\frac{1}{2}}$

Consider the sequence defined recursively by $a_1 = 3$, $a_{n+1} = \frac{a_n}{1+a_n}$.

SCORE: ____ / 3 PTS

- [a] List the first 4 terms of the sequence. Write your final answer as a list.

$$a_2 = \frac{a_1}{1+a_1} = \frac{3}{1+3} = \frac{3}{4}$$

$$a_3 = \frac{a_2}{1+a_2} = \frac{\frac{3}{4}}{1+\frac{3}{4}} \cdot \frac{4}{4} = \frac{3}{7}$$

$$a_4 = \frac{a_3}{1+a_3} = \frac{\frac{3}{7}}{1+\frac{3}{7}} \cdot \frac{7}{7} = \frac{3}{10}$$

$$\boxed{3, \frac{3}{4}, \frac{3}{7}, \frac{3}{10}} \quad \textcircled{2} = \frac{1}{2} \text{ POINT EACH}$$

- [b] Find a formula for the general term a_n of the sequence, assuming that the pattern of the first four terms from [a] continues.

$$a_n = \frac{3}{1+3(n-1)} = \boxed{\frac{3}{3n-2}} \quad \textcircled{1}$$

Determine if $\left\{ \frac{n^2}{\sqrt{n^3+4}} \right\}$ converges or diverges. If it converges, find its limit.

SCORE: ____ / 4 PTS

Justify your answer using proper algebra and/or calculus. State your conclusion clearly.

$$\lim_{n \rightarrow \infty} \frac{n^2}{\sqrt{n^3+4}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{\frac{1}{n} + \frac{4}{n^4}}} = \infty$$

SINCE $\frac{1}{n} + \frac{4}{n^4} \rightarrow 0$ AS $n \rightarrow \infty$

OR

$$= \lim_{n \rightarrow \infty} \frac{n^{\frac{1}{2}}}{\sqrt{1 + \frac{4}{n^3}}} = \infty$$

SINCE $n^{\frac{1}{2}} \rightarrow \infty$ AND $1 + \frac{4}{n^3} \rightarrow 1$ AS $n \rightarrow \infty$

SEQUENCE DIVERGES

Determine if the sequence defined recursively by $a_1 = 2$, $a_{n+1} = \frac{1}{2}(6 + a_n)$ is increasing or decreasing.

SCORE: ____ / 4 PTS

Use mathematical induction to prove your answer. State your conclusion clearly.

$$a_2 = \frac{1}{2}(6+2) = 4$$

$$a_3 = \frac{1}{2}(6+4) = 5$$

$$a_4 = \frac{1}{2}(6+5) = \frac{11}{2}$$

LOOKS LIKE

$\{a_n\}$ IS INCREASING

$$\textcircled{\frac{1}{2}} \underline{a_2 > a_1}, \text{ SINCE } 4 > 2$$

$$\textcircled{\frac{1}{2}} \underline{\text{IF } a_{k+1} > a_k \text{ FOR SOME INTEGER } k \geq 1,}$$

$$\text{THEN } 6 + a_{k+1} > 6 + a_k$$

$$\text{AND } \underline{\frac{1}{2}(6 + a_{k+1}) > \frac{1}{2}(6 + a_k)} \textcircled{1}$$

$$\text{SO } \underline{a_{k+2} > a_{k+1}} \textcircled{1}$$

BY INDUCTION, $a_{n+1} > a_n$ FOR ALL $n \geq 1$

SO $\underline{\{a_n\} \text{ IS INCREASING}} \textcircled{1}$